October Math Gems

Problem of the week 29

§1 Problems

Problem 1.1.

$$\sqrt[4]{1-x^2}+\sqrt[4]{1-x}+\sqrt[4]{1+x}=3$$

Solution.

$$((1-x)(1+x))^{\frac{1}{4}}+(1-x)^{\frac{1}{4}}+(1+x)^{\frac{1}{4}}=3$$

Let,

$$a=(1-x)^{\frac{1}{4}}, \qquad b=(1+x)^{\frac{1}{4}}$$

$$ab+a+b+1=3+1$$

$$(1+b)(1+a)=4 \implies a=1 \quad \text{and} \quad b=1$$

Now, we can say that $1+x=1-x$. So, $x=0$. □

Problem 1.2. Find all points (x,y) where the functions $f(x),g(x),h(x)$ have the same value:

$$f(x)=2^{x-5}+3, \qquad g(x)=2x-5, \qquad h(x)=\frac{8}{x}+10$$

Solution.

$$f(x)=g(x)=h(x)$$

$$2^{x-5}+3=2x-5=\frac{8}{x}+10$$

For,

$$2x-5=\frac{8}{x}+10 \qquad \text{(multiply by } x\text{)}$$

$$2x^2-5x=8+10x \implies 2x^2-15x-8=0$$

$$x=8 \qquad \text{and} \qquad x=\frac{-1}{2}$$

For,

$$2^{x-5}+3=2x-5$$

will give us an integer solution $x=8$. we can verify solution $x=8$ by plugging the value 8 in the place of x. So, we will see that

$$f(x)=g(x)=h(x)=11 \qquad \text{As } (x=8)$$

So, the points that $f(x)=g(x)=h(x)$ have the same value is $(8,11)$. □

Problem 1.3. Solve for x

$$(12x-1)(6x-1)(4x-1)(3x-1)=5$$

Solution. Let $x=\frac{y}{12}$,

$$(12\times\frac{y}{12}-1)(6\times\frac{y}{12}-1)(4\times\frac{y}{12}-1)(3\times\frac{y}{12}-1)=5$$

$$(y-1)(\frac{y}{2}-1)(\frac{y}{3}-1)(\frac{y}{4}-1)=5$$

Multiply this equation by $2\times3\times4$, So we get

$$(y-1)(y-2)(y-3)(y-4)=120$$

Now, we can solve for the four roots by knowing the properties of the 4^{th} degree equation

$$y=\frac{5\pm\sqrt{-39}}{2}\qquad y=6\qquad y=-1$$

Do not forget that we assume before that $x=\frac{y}{12}$. So,

$$x=\frac{5\pm\sqrt{-39}}{24}\qquad x=\frac{1}{2}\qquad x=\frac{-1}{12}$$

□

Problem 1.4. If a,b,c are integers

$$\frac{ab}{a+b}=\frac{1}{3},\qquad\frac{cb}{c+b}=\frac{1}{4},\qquad\frac{ac}{a+c}=\frac{1}{5}$$

Find the value of

$$\frac{24abc}{ab+bc+ca}$$

Solution.

$$\frac{a+b}{ab}=3\implies\frac{a}{b}+\frac{1}{a}=3$$
$$\frac{b+c}{bc}=4\implies\frac{1}{c}+\frac{1}{b}=4$$
$$\frac{ca}{c+a}=5\implies\frac{1}{c}+\frac{1}{a}=5$$

Sum these three equations

$$2(\frac{1}{a}+\frac{1}{b}+\frac{1}{c})=12\implies(\frac{1}{a}+\frac{1}{b}+\frac{1}{c})=6$$

$$\frac{abc}{bc+ac+ab}=\frac{1}{6}$$
$$\frac{24abc}{bc+ac+ab}=\frac{24}{6}=4$$

□

Problem 1.5. The general solution of

$$\sin x-3\sin^2x+\sin^3x=\cos x-3\cos^2x+\cos^3x$$

Solution.

$$(\sin x + \sin 3x) - 3\sin 2x = (\cos x + \cos 3x) - 3\cos 2x$$

$$2\sin 2x \cos x - 3\sin 2x = 2\cos 2x \times \cos x - 3\cos 2x$$

$$\sin 2x(2\cos x - 3) = \cos 2x(2\cos x - 3)$$

$$\sin 2x = \cos 2x \qquad \cos x \neq \frac{3}{2}$$

$$\tan 2x = 1 \implies x = \frac{n\pi}{2} + \frac{\pi}{8} \qquad (\text{As } \frac{\sin 2x}{\cos 2x} = \tan 2x)$$

□

Problem 1.6. Solve the equation

$$[\sqrt{5+\sqrt{24}}]^x - [\sqrt{5-\sqrt{24}}]^x = 40\sqrt{6}$$

Solution.

$$(5+\sqrt{24})(5-\sqrt{24}) = 1$$

multiply the given equation with $(\sqrt{5+\sqrt{24}})^x$, we get

$$(\sqrt{5+\sqrt{24}})^x \times [\sqrt{5+\sqrt{24}}]^x - (\sqrt{5+\sqrt{24}})^x \times [\sqrt{5-\sqrt{24}}]^x = 40\sqrt{6} \times (\sqrt{5+\sqrt{24}})^x$$

$$(\sqrt{5+\sqrt{24}})^{2x} - 1 = 40\sqrt{6} \times (\sqrt{5+\sqrt{24}})^x$$

Suppose that

$$(\sqrt{5+\sqrt{24}})^x = t$$

$$t^2 - 40\sqrt{6} \times t - 1 = 0 \implies t = 20\sqrt{6} \pm 49$$

As,

$$t = 20\sqrt{6} - 49 < 0$$

so it is rejected.

$$t = 20\sqrt{6} + 49 = (5+\sqrt{24})^2 = (\sqrt{5+\sqrt{24}})^x = (5+\sqrt{24})^{\frac{x}{2}}$$

So,

$$\frac{x}{2} = 2 \implies x = 4$$

□

Problem 1.7. Solve

$$x^{[\frac{3}{4}(\log(x))^2 + (\log(x)) - \frac{5}{4}]} = \sqrt{2}$$

Solution.

$$\log_2 x^{[\frac{3}{4}(\log(x))^2 + (\log(x)) - \frac{5}{4}]} = \log_2 (2)^{\frac{1}{2}}$$

$$[\frac{3}{4}(\log_2(x))^2 + (\log_2(x)) - \frac{5}{4}] \times \log_2 x = \frac{1}{2}$$

Let,

$$\log_2 x = y$$

So, we get

$$[\frac{3}{4}(y)^2 + (y) - \frac{5}{4}] \times y = \frac{1}{2} \implies 3y^3 + 4y^2 - 5y - 2 = 0$$

$$3y^3 + 4y^2 - 5y - 2 = (y-1)(3y+1)(y+2) = 0$$

$$y = 1 \qquad y = -2 \qquad y = \frac{-1}{3}$$

Now,

$$\log_2 x = 1 \qquad \log_2 x = -2 \qquad \log_2 x = \frac{-1}{3}$$

So, we will get three values of x,

$$x = 2^1 = 2 \qquad x = 2^{-2} = \frac{1}{4} \qquad x = 2^{\frac{-1}{3}}$$

□

Problem 1.8. If

$$f(n+3) = \frac{f(n)-1}{f(n)+1}, \qquad f(11) = 11$$

Find the value of $f(2003) =$

Solution. Suppose that $n = 11$ so $n + 3 = 14$

$$f(11+3) = f(14) = \frac{f(11)-1}{f(11)+1} = \frac{11-1}{11+1} = \frac{5}{6}$$

Now, we can find $f(17)$ by using the value of $f(14)$

$$f(14+3) = f(17) = \frac{f(14)-1}{f(14)+1} = \frac{\frac{5}{6}-1}{\frac{5}{6}+1} = \frac{-1}{11}$$

$$f(17+3) = f(20) = \frac{f(17)-1}{f(17)+1} = \frac{\frac{-1}{11}-1}{\frac{-1}{11}+1} = \frac{-6}{5}$$

$$f(20+3) = f(20+3) = \frac{f(20)-1}{f(20)+1} = \frac{\frac{-6}{5}-1}{\frac{-6}{5}+1} = 11$$

you must have noticed the pattern, we return to $f(x) = 11$ after 12 rounds, we can suppose that

$$2003 = 11 + 12k$$

and if we get the value of k as an integer number so $f(2003) = 11$. Now, solve for k,

$$2003 = 11 + 12k \implies k = 166$$

So, we can say that $f(2003) = 11$ □

Problem 1.9. Solve for x

$$\frac{2x}{2x^2 - 5x + 3} + \frac{13x}{2x^2 + x + 3} = 6$$

Solution. First, divide the numerator and denominator by x, As $x = 0$ is not one of the roots of this equation. So,

$$\frac{2}{2x - 5 + \frac{3}{x}} + \frac{13}{2x + 1 + \frac{3}{x}} = 6$$

Now, Let

$$a = 2x + \frac{3}{x}$$

$$\frac{2}{a-5} + \frac{13}{a+1} = 6 \implies 13(a-5) + 2(a+1) = 6(a-5)(a+1)$$

$$13a - 65 + 2a + 2 = 6(a^2 - 4a - 5) \implies 6a^2 - 39a + 33 = 0$$

Now, we can solve for a

$$a = \begin{cases} \frac{11}{2} \\ 1 \end{cases}$$

For $a = 1$, there is no real solutions!

For $a = \frac{11}{2}$,

$$(4x-3)(x-2) = 0 \implies x = \frac{3}{4}, \qquad x = 2$$

So, the zeros of this equation is $\{\frac{3}{4}, 2\}$ $\square$

Problem 1.10. If α, β, γ do not differ by a multiple of π and if

$$\frac{\cos(\alpha + \theta)}{\sin(\beta + \gamma)} = \frac{\cos(\beta + \theta)}{\sin(\gamma + \alpha)} = \frac{\cos(\gamma + \theta)}{\sin(\alpha + \beta)} = K$$

Find the value of K.

Solution. Observe that α, β, γ all satisfy the below equation in x

$$\frac{\cos(x + \theta)}{\sin(S - x)} = k \qquad S = \alpha + \beta + \gamma$$

$$\cos x \times \cos\theta - \sin x \times \sin\theta = k \sin S \cos x - k \sin x \cos S$$

$$\sin x(k \cos S - \sin\theta) = \cos x(k \sin S - \cos\theta) \rightarrow (1)$$

Assume that $(k \cos S - \sin\theta) \neq 0$

$$\tan x = \frac{(k \sin S - \cos\theta)}{(k \cos S - \sin\theta)} = \delta$$

So,

$$\tan\alpha = \tan\beta = \tan\gamma = \delta$$

$$\alpha = n\pi + \beta = m\pi + \gamma \qquad \text{(a contradiction)}$$

So,

$$(k \cos S - \sin\theta) = 0$$

and from (1)

$$(k \sin S - \cos\theta) = 0$$

$$\implies k \cos S = \sin\theta \qquad \text{and} \qquad k \sin S = \cos\theta$$

Squaring and adding, we get

$$k^2(\cos^2 S + \sin^2 S) = (\sin^2\theta + \cos^2\theta)$$

$$k^2 = 1 \implies k = \pm 1$$

□

Problem 1.11. If $x^2 + y^2 = 4$, Find the largest value of $3x + 4y$.

Solution.

$$\because (ax+by)^2 + (bx-ay)^2 = (a^2+b^2)(x^2+y^2)$$

$$\therefore (3x+4y)^2 + (4x-3y)^2 = (9+16)(x^2+y^2) = 25 \times 4 = 100$$

$$\because (3x+4y)^2 = 100 - (4x-3y)^2 \qquad \text{and} \qquad (4x-3y)^2 \geq 0$$

$$\therefore |3x+4y| \leq 10 \implies -10 \leq 3x+4y \leq 10$$

so the greatest value of $3x + 4y$ is 10. □

Problem 1.12. If $ax + (b-3) = (5a-1)x + 3b$ has more than one solution, find the value of $100a + 4b$.

Solution.

$$x(a - 5a + 1) = 3b - b + 3 \implies (1-4a)x = 2b + 3$$

We can ask ourselves When the equation has more than one solution?

$$0 \times x = 0 \qquad \text{(x is infinite)}$$

Now, we can use this idea in this proof

$$(1-4b) = 0 \implies a = \frac{1}{4} \qquad 2b + 3 = 0 \implies b = \frac{-3}{2}$$

$$100a + 4b = 100 \times \frac{1}{4} + 4 \times \frac{-3}{2} = 25 - 6 = 19$$

□

Problem 1.13. Find the least value of this algebraic expression

$$\sqrt{x^2+1} + \sqrt{(y-x)^2+4} + \sqrt{(z-y)^2+1} + \sqrt{(10-z)^2+9}$$

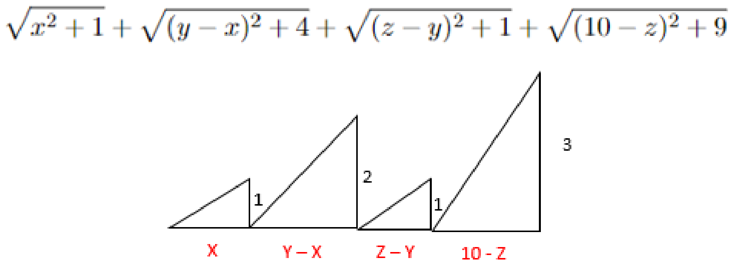
Solution. We can say the least value of this algebraic expression is the hypotenuse of the right-angled-triangle

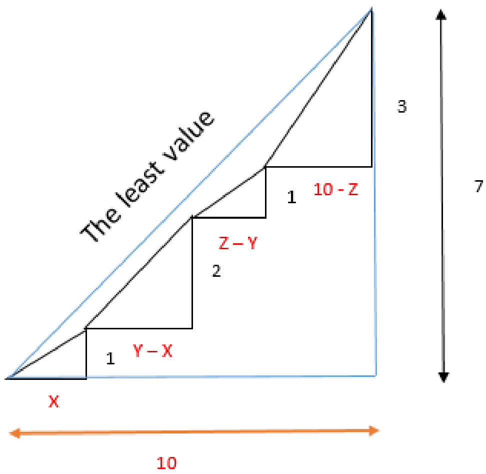
$$\sqrt{(7)^2 + (10)^2} = \sqrt{149}$$

□

Problem 1.14. If $a + b + c = 0$ then the value of

$$\frac{a^7 + b^7 + c^7}{abc(a^4 + b^4 + c^4)}$$





Solution.

$$a+b+c=0 \implies (a+b+c)^2=0$$

$$(a+b+c)^2 = a^2+b^2+c^2+2(ab+ac+bc)=0$$

$$(a^2+b^2+c^2)^2 = (-2(ab+ac+bc))^2 \to *$$

$$(a^2+b^2+c^2)^2 = a^4+b^4+c^4+2(a^2b^2+a^2c^2+b^2c^2) \to (1)$$

$$(-2(ab+ac+bc))^2 = 4[(a^2b^2+a^2c^2+b^2c^2)+2(ab^2c+a^2bc+bc^2a)] = 4[(a^2b^2+a^2c^2+b^2c^2)] \to (2)$$

substitute (1) and (2) in (*), we get

$$a^4+b^4+c^4 = 2(a^2b^2+a^2c^2+b^2c^2) \to (3)$$

$$\because a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ac)=0$$

$$\therefore a^3+b^3+c^3 = 3abc \to (4)$$

Now, multiply (3) and (4)

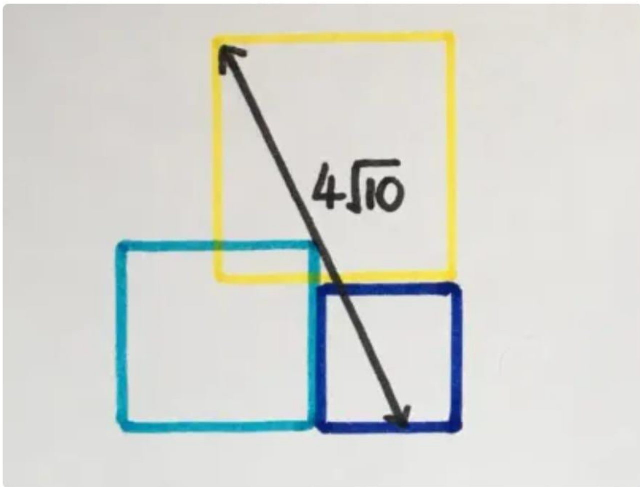
$$(a^4+b^4+c^4)(a^3+b^3+c^3) = a^7+b^c+c^7-abc(a^2b^2+b^2c^2+a^2c^2) = 6abc(a^2b^2+b^2c^2+a^2c^2)$$

$$\therefore a^7+b^c+c^7 = 7abc(a^2b^2+b^2c^2+a^2c^2) \to (5)$$

substitute with (3) and (5) in the needed expression

$$\frac{(a^7+b^7+c^7)}{abc(a^4+b^4+c^4)} = \frac{7abc(a^2b^2+b^2c^2+a^2c^2)}{7abc(a^2b^2+b^2c^2+a^2c^2)} = \frac{7}{2}$$

$\square$



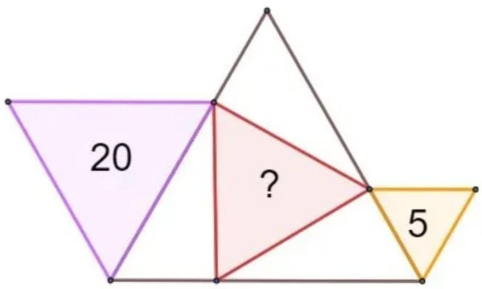
Problem 1.15. The side lengths of the three squares are consecutive integers. What's the total area?

Solution. Note the yellow square is a two-unit wider than the blue one.

$$16 + x^2 = (4\sqrt{10})^2 = 160$$

which means the squares have lengths 5, 6, and 7. □

Problem 1.16. Four equilateral triangles, Find the area of the red one.



Solution.

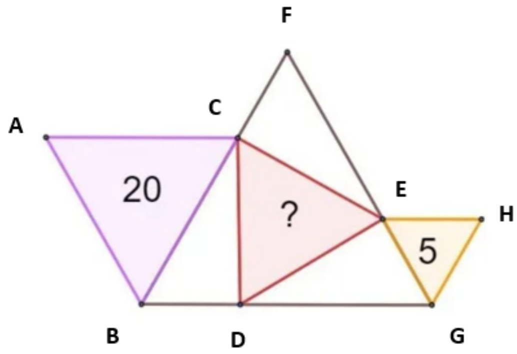
$$\triangle DEC = (1 - 3 \times \frac{1}{3} \times \frac{2}{3})\triangle FBG = \frac{1}{3}\triangle FBG$$
$$\triangle FBG = 20 + 5 \times 5 = 45$$
$$\triangle CDE = \frac{1}{3} \times 45 = 15$$

□

Problem 1.17. If $6^{-z} = 2^x = 3^y$ then the value of

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

is ?



Solution.

$$6^{-z} = 2^x = 3^y = a$$

So,

$$2^x = a \implies 2 = \sqrt[x]{a} \implies 2 = a^{\frac{1}{x}}$$

and so on with others so we get

$$2 = a^{\frac{1}{x}} \qquad 6 = a^{\frac{-1}{z}} \qquad 3 = a^{\frac{1}{y}}$$

We can see that

$$2 \times 3 = a^{\frac{1}{x}} \times a^{\frac{1}{y}} = a^{\frac{1}{x}+\frac{1}{y}} = a^{\frac{-1}{z}}$$

Now, we can say that

$$\frac{1}{x} + \frac{1}{y} = \frac{-1}{z}$$

So,

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{-1}{z} + \frac{1}{z} = 0$$

□

Problem 1.18. Find the solution set of the equation

$$3\cos^{-1} x = \sin^{-1}(\sqrt{1-x^2} \times (4x^2 - 1))$$

Solution.

$$y = \sin^{-1}(\sqrt{1-x^2} \times (4x^2 - 1))$$

$$x = \cos\theta \implies \theta = \cos^{-1} x$$

$$y = \sin^{-1}(\sin\theta(4\cos^2\theta - 1)) = \sin^{-1}(\sin(3\theta)) = 3\theta = 3\cos^{-1}(x) \qquad (\frac{-\pi}{6} \le \theta \le \frac{\pi}{6})$$

also,

$$0 \le \theta \le \frac{\pi}{6} \implies 1 \ge x \ge \frac{\sqrt{3}}{2}$$

So the solution set

$$x \in [\frac{\sqrt{3}}{2}, 1]$$

□

Problem 1.19. Solve

$$x^{x^{x^{2021}}} = 2021$$

Solution. We will find if we put $x^{2021} = 2021$, So

$$x^{2021} = 2021 \implies x = 2021^{\frac{1}{2021}}$$

□

Problem 1.20. if p, q are odd positive numbers since

$$(1 + 3 + 5 + \cdots + p) + (1 + 3 + 5 + \cdots + q) = (1 + 3 + 5 + \cdots + 19)$$

Find the value of $p + q$.

Solution. the rule of the sum of the odd numbers

$$1 + 3 + 5 + \cdots + (2m - 1) = m^2$$

$$2m - 1 = p \implies 2m = p + 1 \implies m = \frac{p + 1}{2}$$

$$\therefore 1 + 3 + 5 + 7 + \cdots + p = (\frac{p + 1}{2})^2 \to (1)$$

$$\therefore 1 + 3 + 5 + 7 + \cdots + q = (\frac{q + 1}{2})^2 \to (2)$$

$$\therefore 1 + 3 + 5 + 7 + \cdots + 19 = (\frac{19 + 1}{2})^2 = 100 \to (3)$$

$$(\frac{q + 1}{2})^2 + (\frac{p + 1}{2})^2 = 100$$

there is one possible solution that is

$$6^2 + 8^2 = 10^2$$

$$p + q = 11 + 15 = 26$$

□
